Higher integrability for solutions to a system of critical elliptic PDE.

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Abstract

We give new estimates for a critical elliptic system introduced by Rivière-Struwe in [13] (see also the work of Rupflin [14] and Schikorra [15]), which generalises PDE solved by harmonic (and almost harmonic) maps from a Euclidean ball $B_1 \subset \mathbb{R}^n$ into Riemannian manifolds. Solutions take the form

$$-\Delta u = \Omega.\nabla u$$

where Ω is an anti-symmetric potential with Ω and ∇u belonging to the Morrey space $\mathcal{L}^{2,n-2}$ making the PDE critical from a regularity perspective (classical theory gives one estimates on ∇u in the weak-Morrey space $\mathcal{L}^{(2,\infty),n-2}$, see Sections A.4 and A.1 for definitions if necessary). We use the Coulomb frame method employed in [13] along with the Hölder regularity already acquired in [14], coupled with an extension of a Riesz potential estimate of Adams [1] in order to attain estimates on $\nabla^2 u \in \mathcal{L}^{s,n-2}$ for any $s < 2$. These methods apply when $n = 2$ thereby re-proving the full regularity in this case (see [17]) using Coulomb gauge methods. Moreover they lead to a self contained proof of the local regularity of stationary harmonic maps in high dimension (see Corollary 1.3).

1 Introduction

We will consider the following system for maps $u \in \mathcal{L}_1^{2,n-2}(B_1)$ (functions with $u, \nabla u$ in the Morrey space $\mathcal{L}^{2,n-2}(B_1, \mathbb{R}^m)$), $\Omega \in \mathcal{L}^{2,n-2}(B_1, so(m) \otimes \wedge^1 \mathbb{R}^n)$ and $f \in L^p(B_1, \mathbb{R}^m)$ for $\frac{n}{2} < p < n$,

$$-\Delta u = \Omega.\nabla u + f, \tag{1}$$

where (1) is satisfied in a weak sense and $B_1 \subset \mathbb{R}^n$ is the unit ball. For the definition of Morrey spaces see Section A.1. The notation $\Omega.\nabla u$ corresponds to both an inner product of one-forms and matrix multiplication, so it reads $(\Omega.\nabla u)^i = \langle \Omega^i_j, \nabla u^j \rangle$ where we sum over j and $\langle , \rangle$ is the inner product of one-forms induced by the Euclidean metric.

Estimating the right hand side of (1) using Hölder's inequality leaves us with Δu in the Morrey space $\mathcal{L}^{1,n-2}$ ($=L^1$ when $n=2$), and the best we can do using singular integral estimates is to conclude that $\nabla u \in \mathcal{L}^{(2,\infty),n-2}$ ($= L^{(2,\infty)}$ when $n=2$). (See Sections A.1 and A.4 for definitions and results if necessary.) These spaces are slightly worse than the spaces we started with, therefore we have lost some information and bootstrapping fails. The anti-symmetry condition on Ω is therefore the key to unlocking hidden regularity of this system as first noticed by Rivière [11].

We could interpret Ω as connection forms for the trivial bundle $B_1 \times \mathbb{R}^m$, for which $\mathrm{d}u \in \Gamma(T^*B_1 \otimes \mathbb{R}^m)$ and the equation reads $\mathrm{d}^*_\Omega(\mathrm{d}u) = f$ where d^*_Ω is the induced covariant divergence given by Ω (the formal adjoint of the covariant exterior derivative, $\mathrm{d}^*_\Omega(\mathrm{d}u) = \mathrm{d}^*\mathrm{d}u + *\Omega \wedge *\mathrm{d}u$). With this more geometric setting it is possible to talk of changes of frame or gauge, and crucially the antisymmetry of Ω means that any change of gauge lies in the orthogonal group and carries a natural L^∞ bound. A change of gauge here is a purely local affair and consists of a map $P : B_1 \to SO(m)$ in which we can express $\mathrm{d}u$, Ω and therefore our PDE. Under this change of gauge the new connection forms Ω_P look like

$$\Omega_P = P^{-1}\mathrm{d}P + P^{-1}\Omega P$$

and we have that

$$\mathrm{d}^*_\Omega(\mathrm{d}u) = P(\mathrm{d}^*_{\Omega_P}(P^{-1}\mathrm{d}u)).$$

Therefore, under a change of gauge, solutions to (1) are also solutions to

$$\mathrm{d}^*_{\Omega_P}(P^{-1}\mathrm{d}u) = P^{-1}f$$

for any such P.

1.1 Two-dimensional domains

Given that the problem here is concerned with improving regularity, the game has been to find a gauge that forces this equation to exhibit nice regularity properties. When $n=2$ it was shown by Tristan Rivière [11] that we can change the gauge such that the term $\Omega.\nabla u$ is effectively replaced by a Jacobian determinant. Thus we may use Hardy space methods (see [2]) or Wente-type estimates ([19]) to improve our situation. It was shown in [11] that the most suitable gauge transform is a small perturbation of the Coulomb gauge (or Uhlenbeck gauge) and in fact it is necessary for the gauge to leave the orthogonal group; moreover these methods allow us to write the PDE as a conservation law.

Solutions with $f \equiv 0$ are shown (also in [11]) to describe critical points of conformally invariant elliptic Lagrangians under some natural growth assumptions (for appropriate Ω). In particular when $f \equiv 0$, (1) describes harmonic maps and prescribed mean curvature equations from Riemannian surfaces into closed, C^2 Riemannian manifolds $N \hookrightarrow \mathbb{R}^m$ isometrically embedded in some Euclidean space.

This PDE has subsequently been studied from a regularity and compactness perspective (see [12], [7], [9], [15], [17]). In [17], it is shown that general solutions to (1) (when $n = 2$) are in $W^{2,s}_{loc}$ for all $s < 2$ by means of a Morrey estimate, and we see that Theorem 1.2 and Proposition 2.1 are the analogues of [17, Theorem 1.1] and [17, Lemma 7.3] in the higher dimensional setting.

1.2 Higher dimensional domains

For $n \geq 3$ Rivière-Struwe [13] showed that we can find a Coulomb gauge in the Morrey space setting (see Section A.5), and that this is enough to conclude partial regularity for general solutions. Again this comes down to the appearance of terms that lie in the Hardy space $\mathcal{H}^1$. It is shown that solutions to (1) describe harmonic (and almost harmonic) maps from the Euclidean ball into arbitrary Riemannian manifolds. As outlined in [13] it would be difficult to carry out the same techniques when $n \geq 3$ as in the case $n = 2$, however Laura Keller [6] has shown that when Ω and ∇u lie in a (slightly more restrictive) Besov-Morrey space, then the methods as in the two dimensional case apply.

The regularity obtained in [13] and [14] is as follows (see also [15]):

Theorem 1.1. *Let u, Ω and f be as in (1). Then there exists $\epsilon = \epsilon(n, m, p)$ such that whenever $\|\Omega\|^2_{\mathcal{L}^{2,n-2}(B_1)} \leq \epsilon$ then $u \in C^{0,\gamma}_{loc}$ where $\gamma = 2 - \frac{n}{p} \in (0, 1)$.*

The optimal Hölder regularity was shown in [14] along with an estimate. To see the optimality just consider the case $\Omega \equiv 0$; we have that $u \in W^{2,p}_{loc} \hookrightarrow C^{0,2-\frac{n}{p}}_{loc}$ when $\frac{n}{2} < p < n$ by Calderon-Zygmund theory and Morrey estimates.

As stated in [13], this theorem allows us to extend the regularity theory for stationary harmonic maps from the Euclidean ball into closed C^2 Riemannian manifolds immersed in some Euclidean space. More precisely it is possible to show that away from a singular set S with $\mathcal{H}^{n-2}(S) = 0$, then any weakly stationary harmonic map is smooth. This follows by a classical theorem stating that continuous weakly harmonic maps are smooth.

1.3 Statement of Results

In this paper we will show improved regularity along with a new estimate when $n \geq 3$. In order to get this estimate we use the Coulomb gauge obtained in [13], Theorem 1.1 and we crucially require an extension of a result of Adams [1] which we prove in Section A.4.

Theorem 1.2. *For $n \geq 2$ let $u \in \mathcal{L}^{2,n-2}_1(B_1, \mathbb{R}^m)$, $\Omega \in \mathcal{L}^{2,n-2}(B_1, so(m) \otimes \wedge^1\mathbb{R}^n)$ and $f \in L^p(B_1)$ ($p \in (\frac{n}{2}, n)$) weakly solve*

$$-\Delta u = \Omega.\nabla u + f$$

Then there exist $\epsilon = \epsilon(n,m,p)$ *and* $C = C(n,m,p)$ *such that whenever* $\|\Omega\|_{\mathcal{L}^{2,n-2}(B_1)} \leq \epsilon$ *then* $\nabla^2 u \in \mathcal{L}^{\frac{2p}{n},n-2}(B_{\frac{1}{2}})$, $\nabla u \in \mathcal{L}^{\frac{2p}{n-p},n-2}(B_{\frac{1}{2}})$ *with*

$$\|\nabla^2 u\|_{\mathcal{L}^{\frac{2p}{n},n-2}(B_{\frac{1}{2}})} + \|\nabla u\|_{\mathcal{L}^{\frac{2p}{n-p},n-2}(B_{\frac{1}{2}})} \leq C(\|u\|_{L^1(B_1)} + \|f\|_{L^p(B_1)}).$$

We see that this generalises [17] to higher dimensions, and that if $\nabla u \in \mathcal{L}^{\frac{2p}{n-p},n-2}$ then $u \in C^{0,\gamma}$ with γ as in Theorem 1.1. An interesting question here is whether the integrability of ∇u can be improved further when $n \geq 3$. One might expect that we should have estimates on ∇u in $L^{\frac{np}{n-p}}$ (consider the case $\Omega \equiv 0$). Clearly the case $n = 2$ is no problem as this gives the (optimal) regularity expected, moreover we have found solutions with $f \equiv 0$ that are not in $W^{2,2}$ or even $W^{2,(2,\infty)}$ (to appear in the author's thesis). Thus we cannot expect that $\nabla u \in L^\infty$ or even $\nabla u \in BMO$ in general. This also explains the range of f that we consider.

An easy consequence of Theorem 1.2 is the following

Corollary 1.3. *For* $n \geq 2$ *let* $u \in \mathcal{L}^{2,n-2}_1(B_1, \mathbb{R}^m)$, $\Omega \in \mathcal{L}^{2,n-2}(B_1, so(m) \otimes \wedge^1 \mathbb{R}^n)$ *weakly solve*

$$-\Delta u = \Omega.\nabla u.$$

For any $q < \infty$ $(s = \frac{2q}{2+q} < 2)$ *there exist* $\epsilon = \epsilon(q,m,n)$ *and* $C = C(q,m,n)$ *such that if* $\|\Omega\|_{\mathcal{L}^{2,n-2}(B_1)} \leq \epsilon$ *then*

$$\|\nabla^2 u\|_{\mathcal{L}^{s,n-2}(B_{\frac{1}{2}})} + \|\nabla u\|_{\mathcal{L}^{q,n-2}(B_{\frac{1}{2}})} \leq C\|u\|_{L^1(B_1)}.$$

Remark 1.4. In the case that $|\Omega| \leq C|\nabla u|$ this automatically gives that when $\|\nabla u\|_{\mathcal{L}^{2,n-2}(B_1)}$ is small enough then $u \in W^{2,q}$ for some $q > n$ yielding $u \in C^{1,\gamma}$ for some $\gamma \in (0,1)$. If we knew that Ω depended on u and ∇u in a smooth way then we could immediately conclude smoothness by a simple bootstrapping argument using Schauder theory. Thus we recover a proof of the regularity of weakly stationary harmonic maps into Riemannian manifolds (away from a singular set S with $\mathcal{H}^{n-2}(S) = 0$). We also mention that passing to the standard local estimates for smooth harmonic maps also easily follows.

Remark 1.5. We remark that Theorem 2.1 and Corollary 1.3 should hold (with some added technicalities) given any smooth metric g on B_1, with u, Ω and f as above weakly solving

$$-\Delta_g u = \langle \Omega, \nabla u \rangle_g.$$

The method we use to prove Theorem 1.2 is (for the most part) broadly the same as that employed in [17], the real difference comes in Section 4 where we obtain a decay type estimate (2) using both Hardy and Morrey space methods.

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2 Proof of Theorem 1.2

We prove Theorem 1.2 based on the following Proposition, analogous to [17, Lemma 7.3], the proof of which is left to Section 3:

Proposition 2.1. *Let $n \geq 2$ and $u \in \mathcal{L}_1^{2,n-2}(B_1, \mathbb{R}^m)$, $\Omega \in \mathcal{L}^{2,n-2}(B_1, so(m) \otimes \wedge^1 \mathbb{R}^n)$ and $f \in L^p(B_1, \mathbb{R}^m)$ for $\frac{n}{2} < p < n$, where $B_1 \subset \mathbb{R}^n$ is the unit ball. Now suppose that u is a weak solution to*

$$-\Delta u = \Omega.\nabla u + f$$

on B_1. Then there exits $\epsilon = \epsilon(n, m, p)$ such that whenever $\|\Omega\|^2_{\mathcal{L}^{2,n-2}(B_1)} < \epsilon$ then $\nabla u \in \mathcal{L}_{loc}^{2,n-2(\frac{n}{p}-1)}(B_1, \mathbb{R}^m)$.

Proof of Theorem 1.2. This proof generalises the ideas needed in the proofs of [17, Lemmata 7.1 and 7.2] to Morrey spaces. Using Proposition 2.1 and the Hölder inequality we see that $\Omega.\nabla u \in \mathcal{L}_{loc}^{1,n(1-\frac{1}{p})}$ since (for appropriate B_R)

$$\begin{aligned} \|\Omega.\nabla u\|_{L^1(B_R)} &\leq \|\Omega\|_{L^2(B_R)} \|\nabla u\|_{L^2(B_R)} \\ &\leq CR^{\frac{n}{2}-1} R^{\frac{n}{2}-\frac{n}{p}+1}. \end{aligned}$$

We can check that the same holds for f (see Section A.7) so by Theorem A.2 (weak estimate) we see that this implies $\nabla u \in \mathcal{L}_{loc}^{(\frac{n}{n-p},\infty),n(1-\frac{1}{p})}$, which in turn (by a scaling argument) gives, for any $\theta < 1$, $\nabla u \in \mathcal{L}_{loc}^{\frac{\theta n}{n-p},n(1-\frac{\theta}{p})}$.

To see this use Section A.7 in order to consider u on $B_R(x_0)$ by $\hat{u}$ on B_1, we have $\|\nabla \hat{u}\|_{L^{\frac{\theta n}{n-p}}(B_1)} \leq C \|\nabla \hat{u}\|_{L^{\frac{n}{n-p},\infty}(B_1)}$ for any $\theta < 1$, then reversing the scaling gives

$$\|\nabla u\|_{L^{\frac{\theta n}{n-p}}(B_R(x_0))} \leq CR^{\frac{n-p}{\theta}-(n-p)} \|\nabla u\|_{L^{\frac{n}{n-p},\infty}(B_R(x_0))}$$

thus

$$\begin{aligned} \|\nabla u\|^{\frac{\theta n}{n-p}}_{L^{\frac{\theta n}{n-p}}(B_R(x_0))} &\leq CR^{n(1-\theta)} \|\nabla u\|^{\frac{\theta n}{n-p}}_{L^{\frac{n}{n-p},\infty}(B_R(x_0))} \\ &\leq CR^{n(1-\theta)} R^{\theta n(1-\frac{1}{p})} \|\nabla u\|^{\frac{\theta n}{n-p}}_{\mathcal{L}^{(\frac{n}{n-p},\infty),n(1-\frac{1}{p})}} \\ &= CR^{n(1-\frac{\theta}{p})} \|\nabla u\|^{\frac{\theta n}{n-p}}_{\mathcal{L}^{(\frac{n}{n-p},\infty),n(1-\frac{1}{p})}}. \end{aligned}$$

The fact that $\nabla u \in \mathcal{L}_{loc}^{\frac{\theta n}{n-p},n(1-\frac{\theta}{p})}$ implies $\Omega.\nabla u \in \mathcal{L}_{loc}^{s,n(1-\frac{s}{p})}$ where $\frac{1}{s} = \frac{1}{2} + \frac{n-p}{\theta n}$, since (for appropriate B_R and $1 = \frac{s}{2} + \frac{s(n-p)}{\theta n}$)

$$\begin{aligned} \|\Omega.\nabla u\|^s_{L^s(B_R)} &\leq \|\Omega\|^s_{L^2(B_R)} \|\nabla u\|^s_{L^{\frac{\theta n}{n-p}}(B_R)} \\ &\leq CR^{\frac{ns}{2}-s} R^{ns(1-\frac{\theta}{p})\frac{n-p}{\theta n}} \\ &= CR^{n(1-\frac{s}{p})}. \end{aligned}$$

We can choose θ so that $s > 1$ but note that we also have $s < \frac{2n}{3n-2p} < \frac{2p}{n}$ for $p \in (\frac{n}{2}, n)$.

We make the following claim:

$$\Omega.\nabla u \in \mathcal{L}_{loc}^{s_k, n(1-\frac{s_k}{p})}, s_k \in (1, \frac{2p}{n}) \Rightarrow$$
$$\Omega.\nabla u \in \mathcal{L}_{loc}^{s_{k+1}, n(1-\frac{s_{k+1}}{p})}, s_k < s_{k+1} \in (1, \frac{2p}{n}).$$

and $s_{k+1} = \frac{2ns_k}{ns_k+2(n-p)}$.

Before we start we can check that $f \in \mathcal{L}^{s_k, n(1-\frac{s_k}{p})}$ with a uniform estimate for any s_k (see Section A.7). Therefore we may apply Theorem A.2 (strong estimate) to yield $\nabla u \in \mathcal{L}_{loc}^{s_k \frac{n}{n-p}, n(1-\frac{s_k}{p})}$, again by Hölder's inequality we have $\Omega.\nabla u \in \mathcal{L}_{loc}^{s_{k+1}, n(1-\frac{s_{k+1}}{p})}$ where $\frac{1}{s_{k+1}} = \frac{1}{2} + \frac{n-p}{s_k n}$ since

$$\begin{aligned}
\|\Omega.\nabla u\|_{L^{s_{k+1}}(B_R)}^{s_{k+1}} &\leq \|\Omega\|_{L^2(B_R)}^{s_{k+1}} \|\nabla u\|_{L^{s_k \frac{n}{n-p}}(B_R)}^{s_{k+1}} \\
&\leq CR^{\frac{n-2}{2}s_{k+1}} R^{n(1-\frac{s_k}{p})\frac{n-p}{s+kn}s_{k+1}} \\
&= CR^{n(\frac{s_{k+1}}{2}+s_{k+1}\frac{n-p}{s_k n})-s_{k+1}-\frac{n-p}{p}s_{k+1}} \\
&= CR^{n-n\frac{s_{k+1}}{p}}.
\end{aligned}$$

We check that

$$\frac{s_k}{s_{k+1}} = \frac{s_k}{2} + \frac{n-p}{n} < \frac{p}{n} + 1 - \frac{p}{n} = 1.$$

If we assume (to get a contradiction) that $s_{k+1} \geq \frac{2p}{n}$ then we have

$$\frac{2ns_k}{ns_k + 2(n-p)} \geq \frac{2p}{n}$$

which implies

$$2ns_k \geq 2ps_k + 2(n-p)\frac{2p}{n}$$

and therefore $s_k \geq \frac{2p}{n}$, a contradiction. Thus the claim holds.

We have the recursive relation $s_{k+1} = \frac{2ns_k}{ns_k+2(n-p)}$, so we have $s_k \uparrow \frac{2p}{n}$ and we have proved that $\Omega.\nabla u \in \mathcal{L}_{loc}^{s, n(1-\frac{s}{p})}$ for all $s < \frac{2p}{n}$. Thus we also have $\nabla u \in \mathcal{L}_{loc}^{s\frac{n}{n-p}, n(1-\frac{s}{p})}$ for all s in this range (see Theorem A.2).

We note here that for $1 < s < t < \frac{2p}{n}$ we have the estimate

$$\|\nabla u\|_{\mathcal{L}^{s\frac{n}{n-p}, n(1-\frac{s}{p})}} \leq C\|\nabla u\|_{\mathcal{L}^{t\frac{n}{n-p}, n(1-\frac{t}{p})}}$$

for $C = C(n,p)$ since

$$\begin{aligned}
\|\nabla u\|_{L^{s\frac{n}{n-p}}(B_R)}^{s\frac{n}{n-p}} &\leq CR^{n(\frac{n-p}{sn}-\frac{n-p}{tn})\frac{sn}{n-p}} \|\nabla u\|_{L^{t\frac{n}{n-p}}(B_R)}^{s\frac{n}{n-p}} \\
&\leq C\|\nabla u\|_{\mathcal{L}^{t\frac{n}{n-p}, n(1-\frac{t}{p})}}^{s\frac{n}{n-p}} R^{n(\frac{n-p}{sn}-\frac{n-p}{tn})\frac{sn}{n-p}} R^{n(1-\frac{t}{p})\frac{s}{t}} \\
&= C\|\nabla u\|_{\mathcal{L}^{t\frac{n}{n-p}, n(1-\frac{t}{p})}}^{s\frac{n}{n-p}} R^{n(1-\frac{s}{p})}.
\end{aligned}$$

Let $s \in (\frac{2p+n}{2n}, \frac{2p}{n})$ and denote by t the next value given in the bootstrapping argument (if $s = s_k$ then $t = s_{k+1}$). Suppose $\nabla u \in \mathcal{L}^{s\frac{n}{n-p}, n(1-\frac{s}{p})}$ giving $\Omega.\nabla u \in \mathcal{L}^{s, n(1-\frac{s}{p})}$. Notice that within this range, by (8) there is a $C = C(n,p)$ independent of s and t such that

$$\begin{aligned}
\|\nabla u\|_{\mathcal{L}^{t\frac{n}{n-p}, n(1-\frac{t}{p})}(B_{\frac{1}{2}})} &\leq C(\|\Omega\|_{\mathcal{L}^{2,n-2}(B_1)} \|\nabla u\|_{\mathcal{L}^{s\frac{n}{n-p}, n(1-\frac{s}{p})}(B_1)} + \|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)}) \\
&\leq C(\|\Omega\|_{\mathcal{L}^{2,n-2}(B_1)} \|\nabla u\|_{\mathcal{L}^{t\frac{n}{n-p}, n(1-\frac{t}{p})}(B_1)} + \|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)}).
\end{aligned}$$

Raising to the power $\mu := t\frac{n}{n-p}$ we see that

$$\|\nabla u\|^{\mu}_{\mathcal{L}^{\mu, n(1-\frac{t}{p})}(B_{\frac{1}{2}})} \leq C(\|\Omega\|^{\mu}_{\mathcal{L}^{2,n-2}(B_1)} \|\nabla u\|^{\mu}_{\mathcal{L}^{\mu, n(1-\frac{t}{p})}(B_1)} + (\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)})^{\mu}).$$

where we can still pick C independent of t since $\mu < \frac{2p}{n-p}$.

Now, rescaling about some ball $B_R(x_0) \subset B_1$ gives us (see Section A.7)

$$\|\nabla \hat{u}\|^{\mu}_{\mathcal{L}^{\mu, n(1-\frac{t}{p})}(B_{\frac{1}{2}})} \leq C(\|\hat{\Omega}\|^{\mu}_{\mathcal{L}^{2,n-2}(B_1)} \|\nabla \hat{u}\|^{\mu}_{\mathcal{L}^{\mu, n(1-\frac{t}{p})}(B_1)} + (\|\hat{f}\|_{L^p(B_1)} + \|\hat{u}\|_{L^1(B_1)})^{\mu}).$$

Undoing the scaling leaves (see Section A.7)

$$\begin{aligned}
R^{\mu + \frac{nt}{p}} \|\nabla u\|^{\mu}_{\mathcal{L}^{\mu, n(1-\frac{t}{p})}(B_{\frac{R}{2}}(x_0))} &\leq C(\|\Omega\|^{\mu}_{\mathcal{L}^{2,n-2}(B_1)} R^{\mu + \frac{nt}{p}} \|\nabla u\|^{\mu}_{\mathcal{L}^{\mu, n(1-\frac{t}{p})}(B_R(x_0))} + \\
&+ (R^{2-\frac{n}{p}} \|f\|_{L^p(B_1)} + R^{-n} \|u\|_{L^1(B_1)})^{\mu}).
\end{aligned}$$

Since $R < 1$, $\mu < \frac{2p}{n-p}$ and $t < \frac{2p}{n}$ we have that

$$\begin{aligned}
\|\nabla u\|^{\mu}_{\mathcal{L}^{\mu, n(1-\frac{t}{p})}(B_{\frac{R}{2}}(x_0))} &\leq C\|\Omega\|^{\mu}_{\mathcal{L}^{2,n-2}(B_1)} \|\nabla u\|^{\mu}_{\mathcal{L}^{\mu, n(1-\frac{t}{p})}(B_R(x_0))} + \\
&+ C(\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)})^{\mu}) R^{-((n+1)\frac{2p}{n-p}+2)}.
\end{aligned}$$

We are now in a position to apply Lemma A.10 for $\Gamma = C(\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)})^{\mu})$, $\epsilon \leq \left(\frac{\epsilon_0}{C}\right)^{\frac{n-p}{2p}}$ and $\epsilon_0 = \epsilon_0(n,k)$ is found for $k = (n+1)\frac{2p}{n-p} + 2$ to give the estimate

$$\|\nabla u\|_{\mathcal{L}^{\mu, n(1-\frac{t}{p})}(B_{\frac{1}{2}})} \leq C(\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)})$$

with C independent of t. We may now pass to the limit $t \uparrow \frac{2p}{n}$ to give

$$\|\nabla u\|_{\mathcal{L}^{\frac{2p}{n-p}, n-2}(B_{\frac{1}{2}})} \leq C(\|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)}).$$

For the second estimate, note that we have $\Omega.\nabla u \in \mathcal{L}^{\frac{2p}{n}, n-2}$ by Hölder's inequality, thus by Theorem A.4 and the proceeding remarks we have finished the proof.

$\square$

3 Proof of Proposition 2.1

We begin with a proposition stating the main decay estimate required, the proof of this is left until Section 4. This decay estimate is analogous to that of part 2. from [17, Theorem 1.5], except that here we crucially require the Hölder regularity already obtained in order to prove (2).

Proposition 3.1. *With the set-up as in Proposition 2.1. Let $\delta > 0$, then there exist $\epsilon = \epsilon(n, m, p) > 0$ small enough and $C = C(\delta, m, n)$ such that when $\|\Omega\|_{\mathcal{L}^{2,n-2}(B_1)} \leq \epsilon$ we have the following estimate ($\gamma = 2 - \frac{n}{p}$)*

$$\|\nabla u\|^2_{L^2(B_r)} \leq C(\delta)(\|\Omega\|^2_{\mathcal{L}^{2,n-2}(B_1)}[u]^2_{C^{0,\gamma}(B_1)} + \|f\|^2_{L^p(B_1)}) + r^n(1+\delta)\|\nabla u\|^2_{L^2(B_1)}. \quad (2)$$

Proof of Proposition 2.1. We follow the argument for the proof of [17, Lemma 7.3]: Pick $\delta = \delta(n, p)$ sufficiently small so that

$$\lambda := \frac{1+\delta}{2^n} < \frac{1}{2^{n-2+2\gamma}} := \Lambda \in \left(\frac{1}{2^n}, 1\right) \quad (3)$$

since $\gamma = 2 - \frac{n}{p} \in (0, 1)$.

Consider the solution on some small ball $B_R(x_0) \subset B_1$. We have (by (2) and Section A.7)

$$\|\nabla \hat{u}\|^2_{L^2(B_r)} \leq C(\|\Omega\|^2_{\mathcal{L}^{2,n-2}(B_1)}[\hat{u}]^2_{C^{0,\gamma}(B_1)} + \|\hat{f}\|^2_{L^p(B_1)}) + r^n\|\nabla \hat{u}\|^2_{L^2(B_1)},$$

and setting $r = \frac{1}{2}$ yields

$$\|\nabla \hat{u}\|^2_{L^2(B_{\frac{1}{2}})} \leq \lambda\|\nabla \hat{u}\|^2_{L^2(B_1)} + C(\|\Omega\|^2_{\mathcal{L}^{2,n-2}(B_1)}[\hat{u}]^2_{C^{0,\gamma}(B_1)} + \|\hat{f}\|^2_{L^p(B_1)}).$$

Undoing the scaling gives

$$\|\nabla u\|^2_{L^2(B_{\frac{R}{2}}(x_0))} \leq \lambda\|\nabla u\|^2_{L^2(B_R)} + CR^{n-2+2\gamma}(\|\Omega\|^2_{\mathcal{L}^{2,n-2}(B_1)}[u]^2_{C^{0,\gamma}(B_R(x_0))} + \|f\|^2_{L^p(B_R(x_0))}).$$

Therefore, setting $R = 2^{-k}$, $k \in \mathbb{N}_0$ and $a_k := \|\nabla u\|^2_{L^2(B_{2^{-k}})}$ we have

$$\begin{aligned} a_{k+1} &\leq \lambda a_k + \left(\frac{1}{2}\right)^{k(n-2+2\alpha)} C(\|\Omega\|^2_{\mathcal{L}^{2,n-2}(B_1)}[u]^2_{C^{0,\gamma}(B_1)} + \|f\|^2_{L^p(B_1)}) \\ &= \lambda a_k + \Lambda^k C(\|\Omega\|^2_{\mathcal{L}^{2,n-2}(B_1)}[u]^2_{C^{0,\gamma}(B_1)} + \|f\|^2_{L^p(B_1)}). \end{aligned}$$

This can be solved to yield (letting $K := C(\|\Omega\|^2_{\mathcal{L}^{2,n-2}(B_1)}[u]^2_{C^{0,\gamma}(B_1)} + \|f\|^2_{L^p(B_1)})$)

$$a_{k+1} \leq \lambda^k a_1 + K\Lambda\frac{(\Lambda^k - \lambda^k)}{\Lambda - \lambda},$$

and by (3), this simplifies to

$$\|\nabla u\|^2_{L^2(B_{2^{-k}}(x_0))} =: a_k \leq C\Lambda^k$$

Thus, for $r \in (0,1/2]$ we have

$$\|\nabla u\|^2_{L^2(B_r(x_0))} \leq Cr^{n-2+2\gamma}.$$

We have

$$\|\nabla u\|^2_{\mathcal{L}^{2,n-2+2\gamma}(B_{\frac{1}{2}})} \leq C.$$

and a covering argument concludes the proof.

□

4 Proof of Proposition 3.1

Proof of Proposition 3.1. We will use the Coulomb gauge in order to re-write our equation, so set ϵ small enough so that we can apply Lemma A.7. We have (see Section A.5 for the relevant background on Sobolev forms)

$$\mathrm{d}^*(P^{-1}\mathrm{d}u) = \langle \mathrm{d}^*\eta, P^{-1}\mathrm{d}u\rangle + P^{-1}f$$

and

$$\mathrm{d}(P^{-1}\mathrm{d}u) = (\mathrm{d}P^{-1} \wedge \mathrm{d}u).$$

We can also set ϵ small enough in order to apply Theorem 1.1 so that $u \in C^{0,\gamma}$ where $\gamma = 2 - \frac{n}{p}$. Now we wish to extend the quantities arising above in the appropriate way: First of all we may extend η by zero. We also extend $P - \frac{1}{|B_1|}\int_{B_1} P$ to $\tilde{P} \in W^{1,2} \bigcap L^\infty(\mathbb{R}^n)$ and finally u to $\tilde{u} \in C^{0,\gamma}(\mathbb{R}^n)$ where each has compact support in B_2 (we may assume $u \in C^{0,\gamma}(\overline{B}_1)$).

Note that we have $\|\nabla \tilde{P}\|_{L^2} \leq C\|\nabla P\|_{L^2(B_1)} \leq C\|\Omega\|_{\mathcal{L}^{2,n-2}(B_1)}$ by Poincaré's inequality and $\nabla \tilde{P} = \nabla P$ in B_1. We also have $\tilde{u} \in C^{0,\gamma}(\mathbb{R}^n)$ with $\|\tilde{u}\|_{C^{0,\gamma}} \leq C\|u\|_{C^{0,\gamma}}$ and (since we may assume $\int u = 0$) we have $\|\tilde{u}\|_{C^{0,\gamma}} \leq C[u]_{C^{0,\gamma}}$, moreover $\tilde{u} = u$ in B_1. All the constants here come from standard extension operators and are independent of the function, see for instance [5].

Now we use Lemma A.9 in order to write $P^{-1}\mathrm{d}u = \mathrm{d}a + \mathrm{d}^*b + h$ with a, b, h as in the Lemma. Notice that we have $\Delta a = \langle \mathrm{d}^*\eta, P^{-1}\mathrm{d}u\rangle + P^{-1}f$ and $\Delta b = \mathrm{d}P^{-1} \wedge \mathrm{d}u$ weakly. We proceed to estimate $\nabla u \in L^2$ by estimating $\|\mathrm{d}a\|_{L^2}$, $\|\mathrm{d}^*b\|_{L^2}$ and using standard properties of harmonic functions in order to deal with $\|h\|_{L^2}$.

We start with $\|\mathrm{d}a\|_{L^2}$; notice that $\langle \mathrm{d}^*\eta, P^{-1}\mathrm{d}u\rangle = \langle \mathrm{d}^*\eta, \mathrm{d}(P^{-1}u)\rangle - \langle \mathrm{d}^*\eta, \mathrm{d}P^{-1}\rangle u = I + II$. For I, pick $\phi \in C_c^\infty(B_1)$ and check (we use that η has zero boundary values)

$$\begin{aligned}
\int *\langle \mathrm{d}^*\eta, \mathrm{d}(P^{-1}u)\rangle\phi &= (\mathrm{d}^*\eta, \mathrm{d}(P^{-1}u)\phi) \\
&= (\mathrm{d}^*\eta, \mathrm{d}(P^{-1}u\phi)) - (\mathrm{d}^*\eta, (\mathrm{d}\phi)P^{-1}u) \\
&= -(\mathrm{d}^*\eta, (\mathrm{d}\phi)P^{-1}u) \\
&\leq \|\nabla\eta\|_{L^2(B_1)}\|\nabla\phi\|_{L^2(B_1)}\|P^{-1}u\|_{L^\infty(B_1)} \\
&\leq C\|\Omega\|_{\mathcal{L}^{2,n-2}(B_1)}\|\nabla\phi\|_{L^2(B_1)}[u]_{C^{0,\gamma}(B_1)}.
\end{aligned}$$

We have $I \in H^{-1}(B_1)$ with

$$\|I\|_{H^{-1}(B_1)} \leq C\|\Omega\|_{\mathcal{L}^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}. \tag{4}$$

For II notice that $\langle \mathrm{d}^*\eta, \mathrm{d}\tilde{P}^{-1}\rangle\tilde{u} = \langle \mathrm{d}^*\eta, \mathrm{d}P^{-1}\rangle u$ in B_1. Moreover $\langle \mathrm{d}^*\eta, \mathrm{d}\tilde{P}^{-1}\rangle \in \mathcal{H}^1(\mathbb{R}^n)$ with

$$\|\langle \mathrm{d}^*\eta, \mathrm{d}\tilde{P}^{-1}\rangle\|_{\mathcal{H}^1} \leq C\|\nabla\eta\|_{L^2(B_1)}\|\nabla\tilde{P}\|_{L^2(B_1)} \leq C\|\Omega\|^2_{\mathcal{L}^{2,n-2}(B_1)}$$

by the results in [2] (see Section A.3). Therefore (see Section A.3) we have $\langle \mathrm{d}^*\eta, \mathrm{d}\tilde{P}^{-1}\rangle\tilde{u} \in h^1(\mathbb{R}^n)$ with

$$\|\langle \mathrm{d}^*\eta, \mathrm{d}\tilde{P}^{-1}\rangle\tilde{u}\|_{h^1(\mathbb{R}^n)} \leq C\|\Omega\|^2_{\mathcal{L}^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}.$$

We also have $\|M_{n-2}(\langle \mathrm{d}^*\eta, \mathrm{d}\tilde{P}^{-1}\rangle\tilde{u})\|_{L^\infty} \leq C\|\Omega\|^2_{\mathcal{L}^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}$ since for $R > 0$

$$R^{2-n}\int_{B_R(x_0)}\langle \mathrm{d}^*\eta, \mathrm{d}\widetilde{P^{-1}}\rangle\tilde{u} = R^{2-n}\int_{B_R(x_0)\bigcap B_1}\langle \mathrm{d}^*\eta, \mathrm{d}\widetilde{P^{-1}}\rangle\tilde{u} \leq C\|\Omega\|^2_{\mathcal{L}^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}$$

(remember η was extended by zero). Now, using the Remark A.6 we have $\langle \mathrm{d}^*\eta, \mathrm{d}\tilde{P}^{-1}\rangle\tilde{u} \in H^{-1}(B_1)$, moreover $\langle \mathrm{d}^*\eta, \mathrm{d}\widetilde{P^{-1}}\rangle\tilde{u} = \langle \mathrm{d}^*\eta, \mathrm{d}P^{-1}\rangle u$ in B_1 so

$$\|II\|_{H^{-1}(B_1)} \leq C\|\Omega\|^2_{\mathcal{L}^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}. \tag{5}$$

Putting (4) and (5) together yields $\langle \mathrm{d}^*\eta, P^{-1}\mathrm{d}u\rangle \in H^{-1}(B_1)$ with (assuming $\epsilon < 1$)

$$\|\langle \mathrm{d}^*\eta, P^{-1}\mathrm{d}u\rangle\|_{H^{-1}(B_1)} \leq C\|\Omega\|_{\mathcal{L}^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}.$$

It is easy to check that $P^{-1}f \in H^{-1}(B_1)$ with $\|P^{-1}f\|_{H^{-1}(B_1)} \leq C\|f\|_{L^p(B_1)}$, overall this means that $a \in W_0^{1,2}(B_1)$ weakly solves

$$\Delta a = \langle \mathrm{d}^*\eta, P^{-1}\mathrm{d}u\rangle + P^{-1}f,$$

so we have

$$\|\nabla a\|_{L^2(B_1)} \leq C(\|\Omega\|_{\mathcal{L}^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)} + \|f\|_{L^p(B_1)}). \tag{6}$$

Now we need to estimate $\|\mathrm{d}^*b\|_{L^2(B_1)}$. We know that $b \in W^{1,2}_N(B_1, \bigwedge^2 \mathbb{R}^n)$ (see Section A.5 for a definition) has $\mathrm{d}b = 0$ and $\Delta b = (\mathrm{d}P^{-1} \wedge \mathrm{d}u)$. We have

$$\|\mathrm{d}^*b\|_{L^2(B_1)} = \sup_{E \in C^\infty(B_1, \bigwedge^1 \mathbb{R}^n)\ \|E\|_{L^2(B_1)} \leq 1} (\mathrm{d}^*b, E).$$

Using a smooth version of Lemma A.9 we can decompose each E by $E = \mathrm{d}e_1 + \mathrm{d}^*e_2 + e_3$ where $e_1 \in C^\infty_0(B_1)$, $e_2 \in C^\infty_N(B_1, \bigwedge^2 \mathbb{R}^n)$ with $\mathrm{d}e_2 = 0$ and $\mathrm{d}e_3 = \mathrm{d}^*e_3 = 0$ (e_3 is a harmonic one form). Notice that $(\mathrm{d}^*b, \mathrm{d}e_1) = 0$ since b has zero normal component and $\mathrm{d}^2 e_1 = 0$. Also we have $(\mathrm{d}^*b, e_3) = 0$ since e_3 is harmonic and b has vanishing normal components. Therefore

$$\begin{aligned}
(\mathrm{d}^*b, E) &= (\mathrm{d}^*b, \mathrm{d}^*e_2) \\
&= (P^{-1}\mathrm{d}u, \mathrm{d}^*e_2) \\
&= (\mathrm{d}(P^{-1}u), \mathrm{d}^*e_2) - ((\mathrm{d}P^{-1})u, \mathrm{d}^*e_2) \\
&= -((\mathrm{d}P^{-1})u, \mathrm{d}^*e_2) \\
&\leq C\|\Omega\|_{\mathcal{L}^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}\|\mathrm{d}^*e_2\|_{L^2(B_1)} \\
&\leq C\|\Omega\|_{\mathcal{L}^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}\|E\|_{L^2(B_1)}.
\end{aligned}$$

Therefore

$$\|\mathrm{d}^*b\|_{L^2(B_1)} \leq C\|\Omega\|_{\mathcal{L}^{2,n-2}(B_1)}[u]_{C^{0,\gamma}(B_1)}. \tag{7}$$

We note here that by [8, Theorem 7.5.1] and $\mathrm{d}b = 0$ that we in fact have the same estimate for ∇b.

We now use the fact that h is harmonic giving that the quantity $r^{-n}\|h\|^2_{L^2(B_r)}$ is increasing, and Lemma A.9 to give

$$\begin{aligned}
\|h\|^2_{L^2(B_r)} &\leq r^n\|h\|^2_{L^2(B_1)} \\
&\leq r^n\|P^{-1}\mathrm{d}u\|^2_{L^2(B_1)} \\
&= r^n\|\mathrm{d}u\|^2_{L^2(B_1)}
\end{aligned}$$

where the last line follows because P is orthogonal.

Going back to our original Hodge decomposition we see that (using Young's inequality, the orthogonality of P, (6) and (7))

$$\begin{aligned}
\|\mathrm{d}u\|^2_{L^2(B_r)} &= \|P^{-1}\mathrm{d}u\|^2_{L^2(B_r)} \\
&\leq (\|h\|_{L^2(B_r)} + \|\mathrm{d}a\|_{L^2(B_r)} + \|\mathrm{d}^*b\|_{L^2(B_r)})^2 \\
&\leq (1+\delta)\|h\|^2_{L^2(B_r)} + C_\delta(\|\mathrm{d}a\|_{L^2(B_r)} + \|\mathrm{d}^*b\|_{L^2(B_r)})^2 \\
&\leq (1+\delta)r^n\|\mathrm{d}u\|^2_{L^2(B_1)} + C_\delta(\|\Omega\|^2_{\mathcal{L}^{2,n-2}(B_1)}[u]^2_{C^{0,\gamma}(B_1)} + \|f\|^2_{L^p(B_1)}).
\end{aligned}$$

This completes the proof.

$\square$

A Background and supporting results

A.1 Morrey, Campanato and Hölder spaces

Our main reference here is [4]. Here we introduce the Morrey spaces $\mathcal{L}^{p,\beta}(E)$ for $1 \geq p < \infty$ and $0 \leq \beta \leq n$ ($E \subset \mathbb{R}^n$). We say that $g \in \mathcal{L}^{p,\beta}(E)$ if

$$M_\beta(|g|^p)(x)^{\frac{1}{p}} := \sup_{r>0} \left(r^{-\beta} \int_{B_r(x) \bigcap E} |g|^p \right)^{\frac{1}{p}} \in L^\infty$$

with norm (which makes $\mathcal{L}^{p,\beta}$ a Banach space)

$$\|g\|_{\mathcal{L}^{p,\beta}(E)} = \|M_\beta(|g|^p)^{\frac{1}{p}}\|_{L^\infty(E)}.$$

Clearly we have $\mathcal{L}^{p,0} = L^p$ and $\mathcal{L}^{p,n} = L^\infty$, also we see that M_n is the usual maximal function. Also note that if we allow $\beta > n$ then $\mathcal{L}^{p,\beta} = \{0\}$. We also omit the obvious inclusions given by Hölder's inequality which are self-evident.

The related Campanato spaces $\mathscr{L}^{p,\beta}$ look similar but make sense for a larger range of β. First of all, for $g \in L^1(E)$ let $g_{r,x} = \frac{1}{|B_r(x) \bigcap E|} \int_{B_r(x) \bigcap E} g$ and we say that $g \in \mathscr{L}^{p,\beta}(E)$ if $g \in L^p(E)$ and

$$[u]_{\mathscr{L}^{p,\beta}(E)} := \sup_{x \in E,\, r>0} \left(r^{-\beta} \int_{B_r(x) \bigcap E} |g - g_{r,x}|^p \right)^{\frac{1}{p}} < \infty$$

with norm (making $\mathscr{L}^{p,\beta}$ Banach spaces)

$$\|g\|_{\mathscr{L}^{p,\beta}(E)} = [g]_{\mathscr{L}^{p,\beta}(E)} + \|g\|_{L^p(E)}.$$

For Lipschitz domains we have $\mathcal{L}^{p,\beta} = \mathscr{L}^{p,\beta}$, when $0 \leq \beta < n$. However $\mathscr{L}^{p,n} = BMO$ for all p as opposed to $\mathcal{L}^{p,n} = L^\infty$. We actually have that $\mathcal{L}^{p,\beta} \subset \mathscr{L}^{p,\beta}$ with a uniform estimate (in n, p and β). The reverse inclusion holds with an estimate whose constant blows up as β approaches n.

Moreover $\mathscr{L}^{p,\beta}$ makes sense for $\beta > n$ and when $n < \beta \leq n + p$ we have $\mathscr{L}^{p,\beta} = C^{0,\gamma}$ with $\gamma = \frac{\beta - n}{p}$. If $\beta > n + p$ then $\mathscr{L}^{p,\beta}$ are the constant functions.

We say that $g \in \mathcal{L}^{p,\beta}_k$ if $g, \nabla^k g \in \mathcal{L}^{p,\beta}$. Using the Poincaré inequality we see that if $g \in \mathcal{L}^{p,\beta}_1$ for some $0 \leq \beta \leq n$ then $g \in \mathscr{L}^{p,p+\beta}$. Therefore if $n - p < \beta \leq n$ then $g \in C^{0,\frac{\beta+p-n}{p}}$. Also the borderline case ($\beta = n - p$) gives $g \in BMO$.

We also introduce here the related weak-Morrey spaces $\mathcal{L}^{(p,\infty),\beta}$, consisting of functions g in the Lorentz space $L^{(p,\infty)}$ or 'weak L^p' with $\sup_{x,\, r>0} r^{\frac{-\beta}{p}} \|g\|_{L(p,\infty)(B_r(x))} \leq \infty$. This condition is equivalent to

$$|\{x \in B_r(x_0) \bigcap B_1 : |g|(x) > s\}| \leq C s^{-p} r^\beta$$

with C independent of x_0 and r.

We will also note the following 'Sobolev-Morrey embedding theorem' [1, Theorem 3.2] generalising the usual Sobolev embedding to Morrey spaces-this is a consequence of Theorem A.2.

Theorem A.1. *Suppose $g \in W^{k,p}$ has $\nabla^k g \in \mathcal{L}^{p,\beta}$ with $0 \leq \beta < n$ and $1 < p < \frac{n-\beta}{k}$, then $g \in \mathcal{L}^{\tilde{p},\beta}$ where $\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{k}{n-\beta}$ and*

$$\|g\|_{\mathcal{L}^{\tilde{p},\beta}} \leq C \|\nabla^k g\|_{\mathcal{L}^{p,\beta}}.$$

Note that setting $\beta = 0$ gives the usual Sobolev embedding for $p > 1$.

A.2 Singular Integrals

We recall here the basics of Elliptic regularity theory on Morrey and Campanato spaces. Define the Newtonian potential operator N on functions $g \in L^1$ by

$$N[f](x) := (\Gamma * g)(x) = \int \Gamma(x-y) g(y)\, dy$$

where $\Gamma(x) = \frac{1}{2\pi} \ln |x|$ when $n = 2$, and $\Gamma(x) = C|x|^{2-n}$ when $n \geq 3$ (for appropriate C). If $g \in C_c^\infty$ then $\Delta N[g] = g$. Writing $w = N[g]$ we have that $N : L^p \to L^p$ is a bounded operator for all $1 \leq p \leq \infty$ for $n \geq 3$. When $n = 2$, $N[g] \in L^p_{loc}$ whenever $g \in L^p$.

We also get $\nabla w = \nabla N[g] = (\nabla \Gamma) * g$, therefore up to a constant we have that $\nabla N[g]$ is convolution by $\frac{x^i}{|x|^n}$ with g. The following result of Adams (see [1] Theorem 3.1 and Proposition 3.2) generalises the standard Lebesge estimates to Morrey spaces.

Theorem A.2. *Let $0 \leq \beta < n-1$ and $1 < p < n - \beta$. Then for $\tilde{p}$ with $\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{1}{n-\beta}$ we have that*

$$\nabla N : \mathcal{L}^{p,\beta} \to \mathcal{L}^{\tilde{p},\beta}$$

is bounded.

When $p = 1$ we have that $(n - \beta > 1)$

$$\nabla N : \mathcal{L}^{1,\beta} \to \mathcal{L}^{(\frac{n-\beta}{n-\beta-1},\infty),\beta}$$

is bounded.

This reduces to well known estimates when $\beta = 0$.

Remark A.3. Re-visiting the proof of Proposition 3.2 and Theorem 3.1 in [1] we see that given a, b with $1 < a \leq p \leq b < n - \beta$ then there is a uniform constant $C = C(a, b)$ such that for $g \in \mathcal{L}^{p,\beta}$ then

$$\|\nabla N[g]\|_{\mathcal{L}^{\tilde{p},\beta}} \leq C \|g\|_{\mathcal{L}^{p,\beta}}$$

for any p and β in this range.

We also have the following result of Peetre [10] which generalises both Calderon-Zygmund and Schauder estimates: Let $g \in \mathscr{L}^{p,\beta}$, $1 < p < \infty$ and $0 \leq \beta < n + p$ and $w = N[g]$. Then $\nabla^2 w = \nabla^2 N[g] = (\nabla^2 \Gamma) * g$ and (as above we see $\nabla^2 N$ as an operator)

Theorem A.4. $\nabla^2 N : \mathscr{L}^{p,\beta} \to \mathscr{L}^{p,\beta}$ *is bounded.*

Therefore we also have for $1 < p < \infty$ and $0 \leq \beta < n$ that $\nabla^2 N : \mathcal{L}^{p,\beta} \to \mathcal{L}^{p,\beta}$ is bounded (see Section A.1).

Now suppose that u weakly solves $\Delta u = g \in \mathcal{L}^{p,\beta}$ on B_1. Then, extending g by zero and letting $w := N[g]$ gives in particular,

$$\|w\|_{L^1(B_1)} + \|\nabla w\|_{\mathcal{L}^{\tilde{p},\beta}(B_1)} + \|\nabla^2 w\|_{\mathcal{L}^{p,\beta}(B_1)} \leq C\|g\|_{\mathcal{L}^{p,\beta}}.$$

Thus we see that $h = w - u$ is harmonic on B_1 and $\|h\|_{L^1(B_1)} \leq C(\|u\|_{L^1(B_1)} + \|g\|_{\mathcal{L}^{p,\beta}})$. Therefore by standard estimates on harmonic functions we have

$$\|\nabla u\|_{\mathcal{L}^{\tilde{p},\beta}(B_{\frac{1}{2}})} + \|\nabla^2 u\|_{\mathcal{L}^{p,\beta}(B_{\frac{1}{2}})} \leq C(\|u\|_{L^1(B_1)} + \|g\|_{\mathcal{L}^{p,\beta}}).$$

Moreover by Remark A.3 with $1 < a \leq p \leq b < n - \beta$ we have $C = C(a, b)$ such that

$$\|\nabla u\|_{\mathcal{L}^{\tilde{p},\beta}(B_{\frac{1}{2}})} \leq C(\|u\|_{L^1(B_1)} + \|g\|_{\mathcal{L}^{p,\beta}}). \tag{8}$$

A.3 Hardy Spaces

Pick $\phi \in C_c^\infty(B_1)$ such that $\int \phi = 1$ and let $\phi_t(x) = t^{-n}\phi(\frac{x}{t})$. For a distribution f we say f lies in the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ if $f_* \in L^1(\mathbb{R}^n)$ where

$$f_*(x) = \sup_{t>0} |(\phi_t * f)(x)|$$

with norm $\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} = \|f_*\|_{L^1(\mathbb{R}^n)}$. Clearly we have the continuous embedding $\mathcal{H}^1(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$. The dual space of $\mathcal{H}^1(\mathbb{R}^n)$ is $BMO(\mathbb{R}^2)$ where $BMO := \{g \in L^1_{loc}(\mathbb{R}^n) : \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |g - \bar{g}| < \infty\}$ (see [3]).

Related to $\mathcal{H}^1$ is the so-called local Hardy space h^1 defined to be those functions for which

$$f_{\tilde{*}}(x) = \sup_{0<t<1} |(\phi_t * f)(x)| \in L^1(\mathbb{R}^n)$$

with corresponding norm. Again we clearly have the continuous embedding $h^1(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$.

By [16, Theorem 1.92] we know that $f \in h^1$ if and only if for any $\varphi \in C_c^\infty$ with $\int \varphi \neq 0$ there is a constant λ such that $\varphi(f - \lambda) \in \mathcal{H}^1(\mathbb{R}^n)$, with

$$\|\varphi(f - \lambda)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C\|f\|_{h^1(\mathbb{R}^n)},$$

where $C = C(\varphi)$ and λ is chosen such that $\int \varphi(f - \lambda) = 0$.

The space $\mathcal{H}^1$ is *not* stable by multiplication of smooth functions since $f \in \mathcal{H}$ implies $\int f = 0$. However for the local Hardy space h^1, as long as the multiplier function is sufficiently regular then we have stability . For instance if $h \in h^1$ and $g \in C^{0,\gamma}$, then $gh \in h^1$ and

$$\|gh\|_{h^1} \leq C(\gamma)\|g\|_{C^{0,\gamma}}\|h\|_{h^1}.$$

We also state here an important result of Coiffman et al, [2] which states that (in particular) given two one forms $D, E \in L^2(\mathbb{R}^n, \bigwedge^1 \mathbb{R}^n)$ such that $\mathrm{d}E = 0$ and $\mathrm{d}^*D = 0$ weakly, then $\langle E, D\rangle \in \mathcal{H}^1(\mathbb{R}^n)$ with

$$\|\langle E, D\rangle\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C\|E\|_{L^2(\mathbb{R}^n)}\|D\|_{L^2(\mathbb{R}^n)}.$$

A.4 Adams Decay

We present here a small extension of a result of Adams [1] giving improved estimates on Riesz potentials acting on functions in some appropriate Morrey space. We state the general theory here as we have not seen the proof of Proposition A.5 elsewhere, although it is really an amalgamation of the proofs of Theorem 1.77 in [16], and Proposition 3.1 in [1]. Readers only interested in how Proposition A.5 affects this paper may consider Remark A.6 and nothing more-this is used in order to obtain estimate (5).

Similar as in [16, Theorem 1.77] we consider $a \in C^\infty(\mathbb{R}^n \backslash \{0\})$ such that a is homogeneous of degree $\alpha - n$ and $|a(x)| \leq C|x|^{\alpha-n}$ and $|\nabla a(x)| \leq C|x|^{\alpha-1-n}$ for all $x \in \mathbb{R}^n\backslash\{0\}$ and some fixed C. Now define the operator

$$A_\alpha(g)(x) = a * g(x) = \int_{\mathbb{R}^n} a(x-y)g(y)dy.$$

We add here that the operator $I_\alpha = a * |g|$ is considered in [1] (which is just A_α for $a(x) = |x|^{\alpha-n}$ applied to $|g|$ instead of g), however the results there trivially go through for more general A_α when applied to g. Another word of warning is that in [1] the Morrey space is defined by replacing 'β' by '$n - \beta$' which explains the apparent discrepancy between our exposition and that of [1], however this is purely notational.

This proposition is a replacement of a weak L^q-estimate given by Proposition 3.2 in [1], which we crucially require in the proof of our main supporting proposition to be a strong estimate. We replace the borderline case $p = 1$ by the Hardy space, thereby giving the strong estimate. We only require the case $\alpha = 1$ here.

Proposition A.5. *Let $0 \leq \beta < n$ and $0 < \alpha < n - \beta$. If $g \in \mathcal{H}^1$ with $M_\beta(g) \in L^\infty$. Then $A_\alpha(g) \in L^{\frac{n-\beta}{n-\beta-\alpha}}$ with*

$$|A_\alpha(g)| \leq C(M_\beta(g)(x))^{\frac{\alpha}{n-\beta}}(g_*(x))^{\frac{n-\beta-\alpha}{n-\beta}},$$

therefore

$$\|A_\alpha(g)\|_{L^{\frac{n-\beta}{n-\beta-\alpha}}} \leq C\|M_\beta(g)\|_{L^\infty}^{\frac{\alpha}{n-\beta}} \|g\|_{\mathcal{H}^1}^{\frac{n-\beta-\alpha}{n-\beta}}.$$

Remark A.6. A corollary of this is that for ($\alpha = 1$,$\beta = n-2$) $g \in \mathcal{L}^{1,n-2} \bigcap h^1(\mathbb{R}^n)$ then $g \in H^{-1}(E)$ for any compact $E \subset \mathbb{R}^n$: Let $\tilde{g} = \psi(g-\lambda) + \psi\lambda$ where $\psi \in C_c^\infty$ and $\psi \equiv 1$ on E (so that $\tilde{g} = g$ in E). We have $\psi(g-\lambda) \in \mathcal{H}^1$ (see Section A.3), and $\lambda\psi \in L^\infty(E)$, so we know $A_1(\tilde{g}) \in L^2(E)$ with ($\tilde{g}$ has the same decay as g)

$$\|A_1(\tilde{g})\|_{L^2(E)} \leq C(\|M_{n-2}(g)\|_{L^\infty(\mathbb{R}^n)} \|g\|_{h^1(\mathbb{R}^n)})^{\frac{1}{2}}.$$

Now set $w = N[\tilde{g}] = \Gamma * \tilde{g}$ where N is the Newtonian potential, we have that $\nabla_i w = \nabla_i N[\tilde{g}] = \nabla_i \Gamma * \tilde{g}$ is an operator of the form A_1 for $a(x) = \frac{x_i}{|x|^n}$ ($\alpha = 1$). Therefore for $\phi \in C_c^\infty(E)$ we can test

$$\int_E g\phi = \int_E (\psi(g-\lambda) + \lambda\psi)\phi = \int_E \Delta w \phi = -\int_E \nabla w . \nabla \phi \leq C\|\nabla w\|_{L^2(E)} \|\phi\|_{W^{1,2}(E)}.$$

Thus

$$\|g\|_{H^{-1}(E)} \leq C(\|M_{n-2}(g)\|_{L^\infty(\mathbb{R}^n)} \|g\|_{h^1(\mathbb{R}^n)})^{\frac{1}{2}}.$$

This is used to obtain estimate (5).

Proof of Proposition A.5. We split $A_\alpha(g)$ up into its near and far parts using a partition of unity subordinate to dyadic annuli of a chosen modulus δ, more precisely: Let $\theta(x) \in C_c^\infty(B_4 \backslash B_{\frac{1}{2}})$ with $\theta(x) > 0$ for $1 \leq |x| \leq 2$. Similarly as is done in Semmes [16] we can arrange so that

$$\sum_{j\in\mathbb{Z}} \theta(\delta^{-1}2^{-j}x) = 1$$

for all $x \in \mathbb{R}^n \backslash \{0\}$, moreover we want for our choice of a that $C\int \theta(\frac{x}{4})a(x) = 1$ for some constant C (for reasons that will become apparant below). Notice that $\theta(4\cdot)a(\cdot) \in C_c^\infty(B_1)$ also.

Now define $\eta^j(x) := \delta^{-1}2^{-j}\theta(\delta^{-1}2^{-j}x)a(x)$. Notice that $\delta 2^j \eta^j(x)$ is the piece of a around $\delta 2^{j-1} \leq |x| \leq \delta 2^{j+2}$, so that

$$A_\alpha(g) = \sum_{j\in\mathbb{Z}} \delta 2^j \eta^j * g = \sum_{j\leq 0} \delta 2^j \eta^j * g + \sum_{j\geq 1} \delta 2^j \eta^j * g = I_{inner} + I_{outer}.$$

The intuition here is that we use the decay estimate we have on g in order to deal with I_{outer} and we use the Hardy space qualities of g in order to deal with I_{inner}.

With that in mind we start with estimating I_{inner}. We use the following claim:

$$|\eta^j * g(x)| \leq C(\delta^{-1}2^{-j})^{1-\alpha} g_*(x).$$

This is easy enough to see, first of all we remark that in our definition of g_* we choose to use the function $\psi(x) := C\theta(\frac{x}{4})a(x)$, therefore $g_*(x) := \sup_{t>0} |\psi_t * g(x)|$.

$$\begin{aligned}
|\eta^j * g(x)| &= \left| \int_{B_{\delta 2^{j+2}}} \delta^{-1}2^{-j}\theta(\delta^{-1}2^{-j}(x-y))a(x-y)g(y)dy \right| \\
&= \left| \int_{B_{\delta 2^{j+2}}} \delta^{-1}2^{-j}\theta(\delta^{-1}2^{-j}(x-y))a(\delta^{-1}2^{-(j+2)}(x-y))g(y)(\delta^{-1}2^{-(j+2)})^{n-\alpha}dy \right| \\
&= C(\delta^{-1}2^{-j})^{1-\alpha}|\psi_{\delta 2^{j+2}} * g(x)| \le C(\delta^{-1}2^{-j})^{1-\alpha} g_*(x).
\end{aligned}$$

We estimate

$$\begin{aligned}
|I_{inner}| &= \sum_{j\le 0} \delta 2^j |\eta^j * g(x)| \\
&\le C\sum_{j\le 0} \delta 2^j (\delta^{-1}2^{-j})^{1-\alpha} g_*(x) \\
&\le C\delta^\alpha g_*(x).
\end{aligned}$$

Now we estimate I_{outer}

$$\begin{aligned}
|I_{outer}| &\le C\sum_{j\ge 1} \int_{\delta 2^{j-1}\le|x-y|\le\delta 2^{j+2}} |\theta(\delta^{-1}2^{-j}(x-y))||a(x-y)||g(y)|dy \\
&\le C\sum_{j\ge 1} \int_{\delta 2^{j-1}\le|x-y|\le\delta 2^{j+2}} |x-y|^{\alpha-n}|g(y)|dy \\
&\le C\sum_{j\ge 1} (\delta 2^{j-1})^{\alpha-n} \int_{|x-y|\le\delta 2^{j+2}} |g(y)|dy \\
&\le C\sum_{j\ge 1} (\delta 2^{j-1})^{\alpha-n} (\delta 2^{j+2})^\beta M_\beta(g)(x) \\
&\le C\delta^{\alpha-(n-\beta)} M_\beta(g)(x).
\end{aligned}$$

Putting together these threads gives us

$$|A_\alpha(g)| \le C(\delta^\alpha g_*(x) + \delta^{\alpha-(n-\beta)} M_\beta(g)(x)).$$

Optimising this over δ gives $\delta = \left(\frac{(n-\beta-\alpha)M_\beta(g)(x)}{\alpha g_*(x)}\right)^{\frac{1}{n-\beta}}$ and finally

$$|A_\alpha(g)| \le C(M_\beta(g)(x))^{\frac{\alpha}{n-\beta}}(g_*(x))^{\frac{n-\beta-\alpha}{n-\beta}}.$$

$\square$

A.5 Hodge Decompositions and Coulomb gauge

We require the following from Riviere-Struwe [13], which shows that we can still find the appropriate Coulomb gauge even in the Morrey space setting.

Lemma A.7. *Let $\Omega \in \mathcal{L}^{2,n-2}(B_1, so(m) \otimes \wedge^1\mathbb{R}^n)$. Then there exists $\epsilon > 0$ such that whenever $\|\Omega\|^2_{\mathcal{L}^{2,n-2}(B_1)} \leq \epsilon$ there exist $P \in W^{1,2}(B_1, SO(m))$ and $\eta \in W^{1,2}_0(B_1, so(m) \otimes \wedge^2\mathbb{R}^n)$ such that $\mathrm{d}\eta = 0$ on B_1 and*

$$P^{-1}\mathrm{d}P + P^{-1}\Omega P = \mathrm{d}^*\eta.$$

Moreover $\nabla P, \nabla\eta \in \mathcal{L}^{2,n-2}(B_1)$ with

$$\|\nabla P\|^2_{\mathcal{L}^{2,n-2}(B_1)} + \|\nabla\eta\|^2_{\mathcal{L}^{2,n-2}(B_1)} \leq C\|\Omega\|^2_{\mathcal{L}^{2,n-2}(B_1)} \leq C\epsilon.$$

Remark A.8. This appears different to the lemma appearing in [13], however upon replacing η with $(-1)^{3n+1} * \xi$ as it appears in [13] we see they are the same. The notation here should be explained, d is the exterior derivative, $\mathrm{d}^* = (-1)^{k(n-k)} * \mathrm{d} *$ is the divergence operator on k-forms (formal adjoint of exterior derivative) and for any form ω, $\nabla\omega$ refers to the collection of all first order derivatives, as opposed to $\mathrm{d}\omega$ or $\mathrm{d}^*\omega$ which refers to new forms comprised of certain combinations of first order derivatives of ω. Of course $*$ is the hodge star operator.

We recall here that there is a natural point-wise inner product for k-forms given by $\langle\omega^1, \omega^2\rangle = *(\omega^1 \wedge *\omega^2)$ and an L^2-inner product given by $(\omega^1, \omega^2) = \int *\langle\omega^1, \omega^2\rangle$.

Our main reference here is [8, Chapter 7] where we can find all of the results stated below, in particular we require the following.

Lemma A.9. *Suppose $\omega \in L^2(B_1, \bigwedge^1 \mathbb{R}^n)$ then there are unique $a \in W^{1,2}_0(B_1)$, $b \in W^{1,2}_N(B_1, \bigwedge^2 \mathbb{R}^n)$ and a harmonic one form $h \in L^2(B_1, \bigwedge^1 \mathbb{R}^n)$ such that*

$$\omega = \mathrm{d}a + \mathrm{d}^*b + h.$$

Moreover $\mathrm{d}b = 0$ with

$$\|a\|_{W^{1,2}(B_1)} + \|b\|_{W^{1,2}(B_1)} + \|h\|_{L^2(B_1)} \leq C\|\omega\|_{L^2(B_1)}$$

and

$$\|\mathrm{d}a\|^2_{L^2(B_1)} + \|\mathrm{d}^*b\|^2_{L^2(B_1)} + \|h\|^2_{L^2(B_1)} = \|\omega\|^2_{L^2(B_1)}.$$

We note here that $W^{1,2}_N(B_1, \bigwedge^k \mathbb{R}^n)$ is the space of forms whose normal boundary part vanishes, which we may define in a trace sense or equivalently for any smooth $k-1$ form ν we have $(\omega, \mathrm{d}\nu) = (\mathrm{d}^*\omega, \nu)$ when $\omega \in W^{1,2}_N(B_1, \bigwedge^k \mathbb{R}^n)$.

Otherwise we have the more general formula for smooth k and $k-1$ forms respectively

$$(\omega, \mathrm{d}\nu) = (\mathrm{d}^*\omega, \nu) + \int_\partial \nu_T \wedge *\omega_N$$

where $_T$ and $_N$ denote the tangential and normal components. (The latter holds for any appropriate Sobolev forms by approximation). Note we could easily define $W^{1,2}_T(B_1, \bigwedge^k \mathbb{R}^n)$ in a weak sense also. We use the following fact: For $a \in W^{1,2}(B_1, \bigwedge^{k-1} \mathbb{R}^n)$, $b \in W^{1,2}(B_1, \bigwedge^k \mathbb{R}^n)$ we have $(\mathrm{d}a, \mathrm{d}^* b) = 0$ if either $a_T = 0$ or $b_N = 0$.

Note that we have $\Delta a = \mathrm{d}^*\omega$ and $\Delta b = \mathrm{d}\omega$ in a weak sense since $\mathrm{d}h = \mathrm{d}^* h = 0$, $\mathrm{d}b = 0$ and since a is a function ($\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$).

A.6 Absorption lemma

We have changed the hypotheses of the following lemma compared to how it appears in [18] and [17], however upon inspection of the proof (which can be found in [18]) it can be checked that the lemma as it is stated here is also proved.

Lemma A.10. *(Leon Simon [18, §2.8, Lemma 2].) Let $B_\rho(y) \subset \mathbb{R}^n$ be any ball, $k \in \mathbb{R}$, $\Gamma > 0$, and let φ be any $[0, \infty)$-valued convex sub additive function on the collection of balls as above in $B_\rho(y)$; thus $\varphi(A) \leq \sum_{j=1}^N \varphi(A_j)$ whenever $A, A_1, A_2, ..., A_N$ are balls in $B_\rho(y)$ with $A \subset \bigcup_{j=1}^N A_j$ and $A \bigcap A_j \neq \emptyset$ for any j. There is $\epsilon_0 = \epsilon_0(k, n)$ such that if*

$$\sigma^k \varphi(B_{\sigma/2}(z)) \leq \epsilon_0 \sigma^k \varphi(B_\sigma(z)) + \Gamma$$

whenever $B_{2\sigma}(z) \subset B_\rho(y)$, then there exists some $C = C(k, n) < \infty$ such that

$$\rho^k \varphi(B_{\rho/2}(y)) \leq C\Gamma.$$

In particular we can apply this lemma when $\varphi(A) = \|k\|^p_{\mathcal{L}^{p,\beta}(A)}$.

A.7 Scaling

We will need to consider u, Ω and f solving

$$-\Delta u = \Omega.\nabla u + f$$

on some small ball $B_R(x_0) \subset B_1$. In order to do so we re-scale $\hat{u}(x) := u(x_0 + Rx)$, $\hat{\Omega}(x) := R\Omega(x_0 + Rx)$ and $\hat{f} := R^2 f(x_0 + Rx)$. First of all we see that

$$-\Delta \hat{u} = \hat{\Omega}.\nabla \hat{u} + \hat{f}$$

on B_1 and we list the scaling properties of the related norms as follows.

- $\|\hat{\Omega}\|_{\mathcal{L}^{2,n-2}(B_1)} = \|\Omega\|_{\mathcal{L}^{2,n-2}(B_R(x_0))}$.
- $[\hat{u}]_{C^{0,\gamma}(B_1)} = R^\gamma [u]_{C^{0,\gamma}(B_R(x_0))}$.
- $\|\hat{u}\|_{L^1(B_1)} = R^{-n} \|u\|_{L^1(B_R(x_0))}$.

- $\|\nabla\hat{u}\|_{\mathcal{L}^{l,\nu}(B_1)} = R^{\frac{l-(n-\nu)}{l}}\|\nabla u\|_{\mathcal{L}^{l,\nu}(B_R(x_0))}$.
- Setting $\nu = 0$ above gives $\|\nabla\hat{u}\|_{L^l(B_1)} = R^{1-\frac{n}{l}}\|\nabla u\|_{L^l(B_R(x_0))}$.
- We also have that the Lorentz spaces $L^{(l,\infty)}$ or 'weak'-L^l scale in the same fashion as the usual L^l spaces,

$$\|\nabla\hat{u}\|_{L^{(l,\infty)}(B_1)} = R^{1-\frac{n}{l}}\|\nabla u\|_{L^{(l,\infty)}(B_R(x_0))}.$$

- $\|\hat{f}\|_{L^p(B_1)} = R^{2-\frac{n}{p}}\|f\|_{L^p(B_R(x_0))}$.
- For $f \in L^p(B_1)$ and $1 \leq s \leq p$ we have

$$\|f\|_{\mathcal{L}^{s,n(1-\frac{s}{p})}(B_1)} \leq C\|f\|_{L^p(B_1)}$$

for $C = C(n,p)$, since by Hölder's inequality

$$\begin{aligned} \|f\|^s_{L^s(B_R)} &\leq CR^{n(\frac{1}{s}-\frac{1}{p})s}\|f\|^s_{L^p(B_1)} \\ &= CR^{n(1-\frac{s}{p})}\|f\|^s_{L^p(B_1)}. \end{aligned}$$

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